Lect 3. Topological QPT.

I. The context

□ Caveats of $d \to d+z$ mapping

∘ $d \to d+z$

- the "d" side (classical field theory)

GL free energy functional (coarse grained effective Hamiltonian)

$$-\ln \mathcal{Z} \equiv \beta \mathcal{H}[\Psi] = \int d^d\vec{x}\, \frac{K}{2}|\underbrace{\vec{\nabla}\Psi}_{\text{static fluct}}|^2 + \frac{t}{2}|\Psi|^2 + u|\Psi|^4$$

: No dynamics

- the "$d+z$" side

$$\underbrace{[\hat{\Psi}(x), \hat{\Psi}^\dagger(x')]}_{\text{conjugate mom for } \Psi\text{, i.e. } \Pi_\Psi} = i\hbar\,\delta(x-x')$$

$$S_E[\Psi,\Psi^*] = \int d\tau \int d^d\vec{x}\; \underbrace{\Psi^*\partial_\tau\Psi}_{\text{like } p\dot{q}} \underbrace{+}_{\tau \text{ is imaginary}} \frac{K}{2}|\vec{\nabla}\Psi|^2 + \frac{t}{2}|\Psi|^2 + u|\Psi|^4$$

$d \to d+z$ for the thermodynamic properties of the QCP

$\Leftrightarrow$ treat the R.H.S of Eq (2) like Eq (1) in higher dim.

- Caveats.
  - mapping holds for thermodynamics only
  - resulting classical system can be unusual and anisotropic ($z \neq 1$)
  - extra complication w/ no classical counter part may arise
    e.g. Berry phases, sign problem
  - there are QPT's not involving O.P.
    e.g. Level crossing without symmetry change

Topo order

- Conventional order vs Topo. order

Conventional order	Topo. order
- degeneracy in the GS Hilbert space.	
1) degenerate states are related by local symmetry operation $\uparrow\uparrow\uparrow\uparrow \Rightarrow \downarrow\downarrow\downarrow\downarrow$	1) degeneracy depends on the topology of the entire system. 2) States are related through large loops 
- can be detected through local O.P.? O	X
- Framework for studying QPT?	

☐ An exactly solvable model

1) (infinitely) many conserved quantities
   ↳ operators that commute with $\hat{H}$

2) could be artifical

3) the knowledge of the entire spectrum is precious

   ← more than a Hamiltonian reverse engineered from a desired ground state wave function
   i.e. ① Design $|\psi\rangle$ → ② Design $\hat{H}$ such that $\hat{H}|\psi\rangle = 0$

   ⇒ can open doors to finite T phase diagram

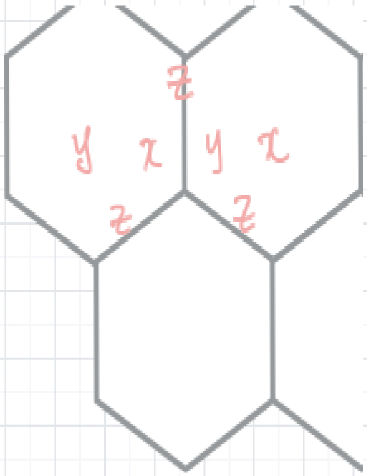
4) can gain some insight into how to think about cases belonging to the same "universality class".

⇒ Reasons for working out with the transverse field Ising model when first learning conventional QPT.

II. Topological QPT in the CSL model (in prep, S.-B Chung, H. Yao, T.L. Hughes, EAK)

□ The Model.

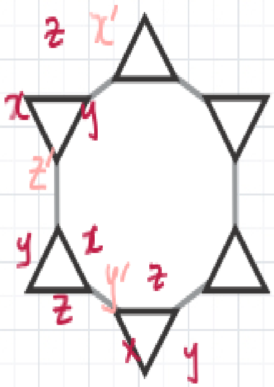
- o the Kitaev model (Ann. Phys <u>321</u>, 2 (2006))



$$\mathcal{H} = \underset{\alpha=(x,y,z)}{\text{Sum over}}\left[\sum_{\langle i,j\rangle \in \alpha\text{-link}} J_\alpha \sigma_i^\alpha \sigma_j^\alpha - \sum_i h_\alpha \sigma_i^\alpha \right] \qquad \alpha = x, y, z$$

- A "spin model" with very anisotropic exchange interaction
- Exactly solvable for $h=0$.
- Gapped phase turn non-Abelian for $h \neq 0$ (perturbative)

- o CSL model (Yao-Kivelson PRL <u>99</u>, 247203 (2007))

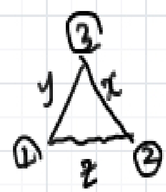


- Decorate vertices with triangles

$$\mathcal{H} = \sum_\alpha \Big(\sum_{\alpha\text{-links}} J \sigma_i^\alpha \sigma_j^\alpha \Big) + \sum_{\alpha'} \Big(\sum_{\alpha'\text{-links}} J' \sigma_i^{\alpha'} \sigma_j^{\alpha'} \Big)$$

- GS always break TRS

$\because$ Define $\phi_\Delta \equiv \sigma^x_{1} \sigma^y_{2} \sigma^z_{3}$



$[\hat{\mathcal{H}}, \hat{\Phi}_p] = [\hat{\Phi}_p, \hat{\Phi}_{p'}] = 0$ $\quad$ Φ_p's are conserved quantities

$\phi_\Delta \xrightarrow[TRS]{} -\phi_\Delta$ $\quad \because$ triangle

□ Solving CSL model using Majorana fermion rep. of quantum spins.

- Majorana fermion rep. $\sigma_j^\alpha = i\, c_j\, d_j^\alpha$

where $\{c_i, c_j\} = 2\delta_{ij}$ $\{d_i^\alpha, d_j^\beta\} = 2\delta_{ij}\delta_{\alpha\beta}$

$c_i^\dagger = c_i$, $d_i^{\alpha\dagger} = d_i^\alpha$

Contrast to usual (complex) fermion $\{a_i^\dagger, a_j\} = \delta_{ij}$

under constraint $d_j^x d_j^y d_j^z c_j = -1$ $\because$ $\sigma_j^x \sigma_j^y \sigma_j^z = i$ for spin 1/2

$$H[\{\hat{u}_{ij}\}] = J \sum_{\alpha\text{-link}} \hat{u}_{ij}\, i c_i c_j + J' \sum_{\alpha'\text{-link}} \hat{u}_{ij}\, i c_i c_j$$

$\hat{u}_{ij} \equiv -i d_i^\alpha d_j^\alpha$ for the given link, $[\hat{u}_{ij}, \hat{H}] = 0$, $[\hat{u}_{ij}, \hat{u}_{kl}] = 0$

takes ± 1 values. $\rightarrow$ replace $\hat{u}_{ij}$'s with $\{u_{ij} = \pm 1\}$

$\Rightarrow$ A quadratic Hamiltonian

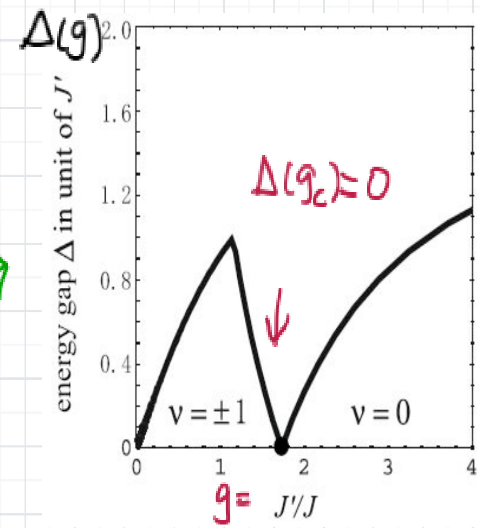
$\Rightarrow$ Diagonalize $\mathcal{H}[\{u_{ij}\}] = \sum_{n,\vec{k}} \epsilon_{n,\vec{k}}[\{u_{ij}\}; g]\, (b^\dagger_{n,\vec{k}} b_{n,\vec{k}} - 1/2)$, (3)

- Turns out vortices always (for all g) cost finite energy

or

- Focus on uniform flux state

at low energy. Vortex free spectrum has a gap $\Delta(g)$



☐ A change in the topological degeneracy at the topo. QPT.

- Count only the states that are physical for the quantum spin model
  - ⇒ a constraint for Majorana fermions $d_i^x d_i^y d_i^z c_i = -1$
  - ⇒ Use a projector to annihilate unphysical states

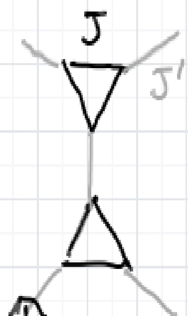
$$\hat{P} \equiv \prod_i \left(\frac{1 - d_i^x d_i^y d_i^z c_i}{2} \right)$$

- The effect of projector can depend on model parameters
  - ⇒ Can change the physical part of the spectrum

- pre-projection : Four inequivalent global flux states are degenerate

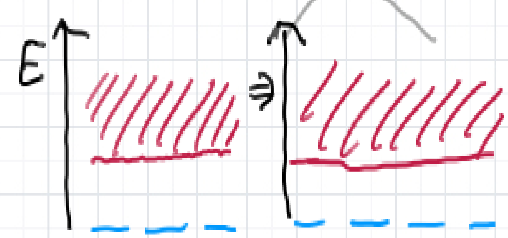
  post-projection :

$g \equiv J'/J$



	$g < g_c$	$g > g_c$
vortex statistics	Non-Abelian	Abelian (Boson/Semion)
G. state deg on a torus	3	4

A topo QPT



spectrum change

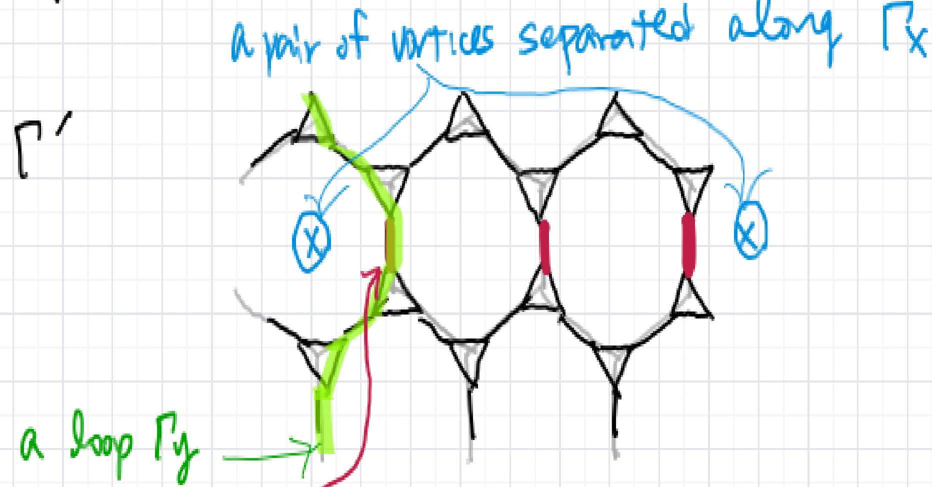
- Same "universality class"

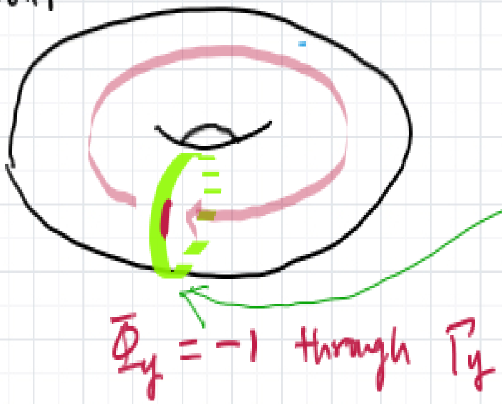
  - weak (BCS) to strong (BEC) equal spin paired p+ip superfluid (at the mean-field BdG Hamiltonian level)
  - Moore-Read state for $\nu = 5/2$ Quantum Hall state on the non-Abelian side
  - Honeycomb lattice Kitaev model under external field
  - toric code on the Abelian side

vortex statistics	non-Abelian	Abelian (Boson/Semion)
G. State deg on a torus	3	4
CSL model	$g < g_c$	$g > g_c$
ESP p+ip SF (BdG)	$\mu < 0$ (BCS)	$\mu > 0$ (BEC)
$\nu = 5/2$ QH	Moore-Read	
Kitaev model $h \neq 0$	$h >$	$h <$
Z_2 matter + gauge		Kitaev toric code

Threading global flux.

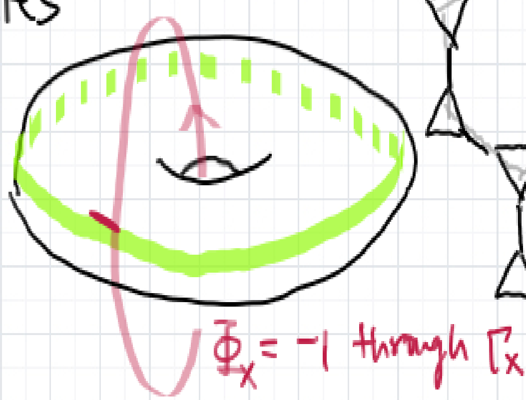
- A global flux can be threaded through a large loop Γ.
  - i) create a pair of vortices
  - ii) transport along Γ' by flipping bonds $\perp \Gamma'$
  - iii) annihilate the pair

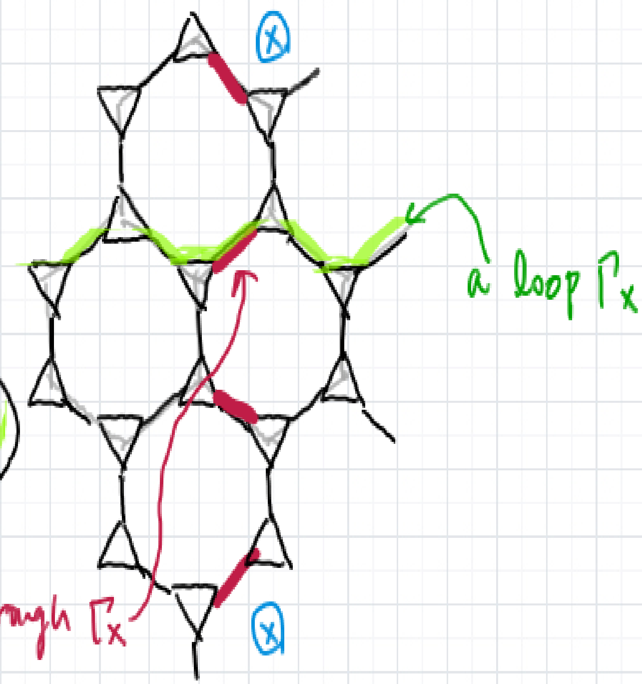




- Leaves the system in the uniform flux state
  $\Rightarrow$ no energy cost $\rightarrow$ Degenerate Ground state.
- Four inequivalent global flux states

$$(\Phi_x, \Phi_y) = (1,1), (1,-1), (-1,1), (-1,-1)$$





- Global flux expectation value

  - Finding physical states for each (Φ_x, Φ_y) config

    ① Diagonalize $\mathcal{H}[\{u_{ij}\}]$ for the given $\{u_{ij}\}$

    ← requires finding the complex fermion eigenstates $b_{n,\vec{k}}$

    diagonalizing a 6x6 matrix. $\mathcal{H}(\vec{k}, \{u_{ij}\})$

    ⇒ obtain the spectrum $\varepsilon_n(\vec{k})$, $n=1,2,3$ for complex fermions $b_{n\vec{k}}$

    ② Many body eigenstates for given $\{u_{ij}\}$

    : occupy or leave out each single particle states

    ③ Hit the many body eigenstates with the projector

$$\hat{P}|\psi\rangle = \begin{cases} 0 & \leftarrow \text{unphysical} \\ |\psi\rangle & \leftarrow \text{physical} \end{cases}$$

  - Ground state degeneracy can be affected by projection

    ∵ $\hat{P}$ kills states with definite fermion parity, even/odd depends on (Φ_x, Φ_y)

because

$$2^N \hat{P} = [1 + \prod_i (-d_i^x d_i^y d_i^z c_i)][1 + \sum_j (-d_j^x d_j^y d_j^z c_j) + \cdots]$$

$$\propto \left[1 + \underbrace{(\prod_i i c_{2i-1} c_{2i})}(\underbrace{\prod_{mn} \hat{U}_{mn}})\right], \qquad (4)$$

using $(d_i^x d_i^y d_i^z c_i)^2 = 1$

- With the change of spectrum with tuning g through g_c,

if $\hat{P}$ kills odd states for all (Φ_x, Φ_y) for $g > g_c$	odd states for $(1,1)$, $(1,-1)$, $(-1,1)$ even states for $(-1,-1)$ for $g < g_c$

Includes G.S. with zero occupation.

- Define global flux expectation value

$$\langle \Phi_\alpha(T) \rangle \equiv \frac{1}{Z} \mathrm{tr}\ \Phi_\alpha e^{-\mathcal{H}/T}. \tag{6}$$

Clearly at $T=0$,

$$\langle \Phi_x \rangle (T=0) = \begin{cases} 1/3 & (\mathrm{nA}, g < g_c) \\ 0 & (\mathrm{A}, g > g_c) \end{cases}. \tag{8}$$

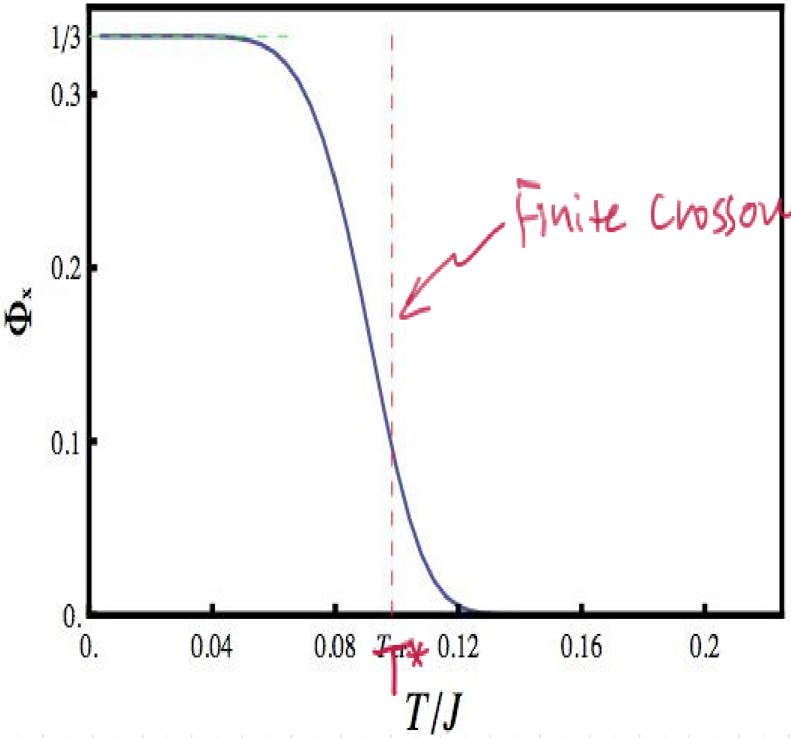
Notice the expectation value is related to the topological degeneracy

$$n_{DEG} = 4 - 3\langle \Phi_x \rangle (T=0). \tag{9}$$

☐ Finite temperature crossover.

- Considering that vortex gap remains finite throughout, focus on uniform flux sectors near critical point.

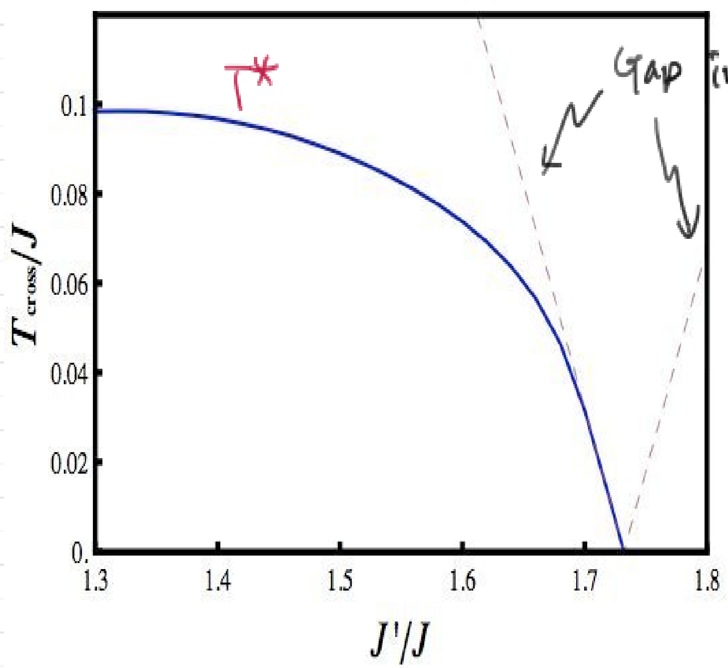
$$\langle \Phi_\alpha \rangle = \frac{\sum_{\Phi_x,\Phi_y=\pm 1} \Phi_\alpha \mathcal{Z}^{(\Phi_x,\Phi_y)}}{\mathcal{Z}}, \qquad (7)$$



Finite crossover temperature scale T^*

We can define T^* through the exponential decay of $\Phi_x(T)$

- T^* turns on upon $g < g_c$ (Non-Abelian side).



Gap in the uniform flux state spectrum.

: shows both side of topo QPT has finite gap.

Both has topo. degeneracy

⇒ This is a QPT between two distinct topo phases

- Finite size dependence

$$T^* \sim \frac{\Delta(g)}{\ln N}, \qquad (10)$$

: Decays in thermodynamic limit, but slower than any other quantity in the problem.

Why $\propto 1/\ln N$? ⇐ You can show $\langle \Phi_x \rangle \gtrsim \left(\tanh \Delta(g)/2T\right)^N$

in the limit, $T \ll \Delta$, N large, show $T^* \propto 1/\ln N$

□ Open Questions

- Do we expect any power law in $T^*(g)$?
- Do these observations bear implications beyond this particular case we studied?
- Though in this case we know why $T^* \propto 1/\ln N$, is this something to be expected in general in 2D topo order?

The manuscript is under preparation. by

S.-B. Chung, H. Yao, T. Hughes, EAK.

① If you want to know more detail, stay tuned to arXiv.

② If you have good idea about above questions, email me!!

(or write a paper on it. ☺)